# Identification Analysis for Randomized Experiments with Noncompliance and "Truncation-by-Death"*

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**Abstract**

Zhang and Rubin (2003) derives the bounds on the average treatment effect in randomized experiments with "truncation-by-death." Imai (Forthcoming) proves that these bounds are sharp and derives the sharp bounds on the quantile treatment effect. In this paper, I extend their results to randomized experiments with noncompliance and derive the sharp bounds on the treatment effects for compliers.

**Key Words:** Average treatment effect, Causal inference, Encouragement design, Complier average causal effect, Principal stratification.

## 1 Introduction

"Truncation-by-death" refers to the problem where some quality-of-life outcome measures are unavailable for patients who die during randomized clinical trials. In such situations, standard missing data analysis techniques, e.g., imputation and weighting, are not applicable because the outcome variables are not even defined for diseased patients. Similar problems also arise in other applied research. Using the principal stratification framework of Frangakis and Rubin (2002), Zhang and

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Rubin (2003) derives the bounds on the average treatment effect. Applying and extending the analytical results of Horowitz and Manski (1995), Imai (Forthcoming) proves that these bounds are sharp and derives the sharp bounds on the quantile treatment effect.

In this paper, I extend their results to randomized experiments with noncompliance and derive the sharp bounds on the treatment effects for compliers. The rest of the paper is organized as follows. In Section 2, I review the statistical framework of randomized experiments with noncompliance, define the estimands, and describe the basic assumptions. Section 3 derives the sharp bounds on the causal effects of interest under the exclusion restriction for noncompliers. In Section 4, I show how to tighten these bounds by imposing additional assumptions. Finally, Section 5 concludes.

## 2 Framework and Assumptions

I first review the statistical framework of randomized experiments with noncompliance and introduce the notation and assumptions used throughout the paper (see Angrist, Imbens, and Rubin, 1996). Consider a random sample of size $N$ from a population. In an "encouragement design," the encouragement to receive the treatment rather than the treatment itself is randomized. Let $Z_i$ be a randomized encouragement indicator variable which is equal to 1 if unit $i$ is encouraged to receive the treatment and is equal to 0 otherwise. I use $T_i(z)$ to denote the binary *potential* treatment indicator variable under the encouragement status $Z_i = z$ for $z = 0, 1$. For example, $T_i(1)$ represents the treatment status of unit $i$ if it is encouraged to receive the treatment. Then, the observed treatment variable can be written as $T_i = Z_i T_i(1) + (1 - Z_i) T_i(0)$. In the framework introduced by Angrist *et al.* (1996), there are four types of units based on the possible values of the potential treatment variables; *compliers* $((T_i(1), T_i(0)) = (1, 0))$, *always-takers* $((T_i(1), T_i(0)) = (1, 1))$, *never-takers* $((T_i(1), T_i(0)) = (0, 0))$, and *defiers* $((T_i(1), T_i(0)) = (0, 1))$.

Since the outcome variable is truncated "by death" for some units, I introduce the potential truncation indicator variable, $W_i(z)$ for $z = 0, 1$ which is equal to 1 if the outcome of unit $i$ would be truncated by death under the encouragement status and is equal to 0 otherwise. The observed "truncation-by-death" variable is given by $W_i = Z_i W_i(1) + (1 - Z_i) W_i(0)$. Note that the outcome variable, $Y_i$, is only observed if and only if $W_i = 0$. Indeed, in the problem of "truncation-by-death" described above, the outcome variable is not even defined for units $W_i = 1$ (Zhang and Rubin, 2003). Hence, if the potential outcome variable is denoted by $Y_i(W_i(z), z)$ for $z = 0, 1$, then $Y_i(0, z)$ is defined while $Y_i(1, z)$ is not. Thus, the "truncation-by-death" problem implies that the causal effect of the randomized encouragement $Z$ is defined only for those units with $W_i(1) = W_i(0) = 0$ (Zhang and Rubin, 2003). Thus, the quantities of interest include the ITT effect, $\tau_{ITT} \equiv E[Y_i(1) - Y_i(0) \mid W_i(0) = 0, W_i(1) = 0]$, as well as the Complier Average Treatment Effect (CATE) defined as,

$$\tau_{CATE} \equiv E[Y_i(1) - Y_i(0) \mid W_i(0) = 0, W_i(1) = 0, C_i = c], \tag{1}$$

Compliers are the subpopulation of units who would receive the treatment only if they are encouraged. Angrist *et al.* (1996) shows that in the absence of the "truncation-by-death" problem, the CATE can be identified using the instrumental variable method.

Given these notations, the randomization of the encouragement ensures that the following assumption holds,

Assumption 1 (Randomization of the Encouragement)

$$Z_i \perp\!\!\!\perp \{T_i(0), T_i(1), W_i(0), W_i(1), Y_i(0,0), Y_i(0,1)\}.$$

Throughout the paper, I make the following additional assumption introduced by Angrist *et al.* (1996),

Assumption 2 (No Difier)

$$T_i(1) \geq T_i(0), \quad \text{for all } i.$$

| Observed Strata $(W_i, T_i, Z_i)$ | Principal Strata $(C_i, W_i(0), W_i(1))$ |
|---|---|
| $(0,0,0)$ | $(n,0,0)$, $(n,0,1)$, $(c,0,0)$, $(c,0,1)$ |
| $(0,1,0)$ | $(a,0,0)$, $(a,0,1)$ |
| $(1,0,0)$ | $(n,1,0)$, $(n,1,1)$, $(c,1,0)$, $(c,1,1)$ |
| $(1,1,0)$ | $(a,1,0)$, $(a,1,1)$ |
| $(0,0,1)$ | $(n,0,0)$, $(n,1,0)$ |
| $(0,1,1)$ | $(a,0,0)$, $(a,1,0)$, $(c,0,0)$, $(c,1,0)$ |
| $(1,0,1)$ | $(n,0,1)$, $(n,1,1)$ |
| $(1,1,1)$ | $(a,0,1)$, $(a,1,1)$, $(c,0,1)$, $(c,1,1)$ |

Table 1: Observed Strata and Principal Strata in Randomized Experiments with Noncompliance and Truncated Outcomes Under the Assumption of No Defier (Assumption 2).

Under Assumption 2, the observed strata, which are defined by the values of the completely observed random variables $(W_i, T_i, Z_i)$, can be written as a mixture of the principal strata (Frangakis and Rubin, 2002), which are defined by the partially observed random variables $(C_i, W_i(0), W_i(1))$, as shown in Table 1.

## 3 Sharp Bounds under Exclusion Restrictions for Noncompliers

In this section, I derive the sharp bounds on the CATE under exclusion restrictions. First, I consider the following exclusion restriction on the truncation for noncompliers,

Assumption 3 (Exclusion Restriction for Noncompliers)

$$W_i(1) \;=\; W_i(0) \quad \text{if} \quad C_i \in \{a, n\}.$$

The assumption states that for always-takers and never-takers the values of the potential truncation-by-death indicator variables are the same regardless of their encouragement status, $Z_i$. Next, define the following quantities,

$$p_{jtz} \;\equiv\; \Pr(W_i = j, T_i = t \mid Z_i = z), \tag{2}$$

$$\pi_{sjk} \;\equiv\; \Pr(C_i = s, W_i(0) = j, W_i(1) = k). \tag{3}$$

for $j, k, t, z \in \{0, 1\}$ and $s \in \{c, a, n\}$. Then, Assumption 3 implies that $\pi_{n01} = \pi_{n10} = 0$ and $\pi_{a01} = \pi_{a10} = 0$. Appendix A.1 proves that under Assumption 3, the sharp bounds on $\pi_{c00}$ are given by,

$$\pi_{c00} \in \Pi \equiv (0,1] \bigcap \left[p_{011} + p_{101} - p_{010} - p_{100}, \ \min(p_{011} - p_{010}, p_{000} - p_{001})\right]. \tag{4}$$

with the following testable implications,

$$p_{000} \geq p_{001}, \quad p_{100} \geq p_{101}, \quad p_{011} \geq p_{010}, \quad \text{and} \quad p_{111} \geq p_{110}. \tag{5}$$

Denote the distribution of each potential outcome $Y(0, z)$ for units with $(C_i, W_i(1), W_i(0)) = (s, j, k)$ as $P_{sjk|z}$, and define the distribution of each observed outcome $Y_i$ for units with $(T_i, Z_i) = (t, z)$ as $P_{tz}$. Then, under Assumption 3, the following relationships hold,

$$\frac{p_{000}P_{00} - p_{001}P_{10}}{p_{000} - p_{001}} = \frac{\pi_{c00}}{p_{000} - p_{001}} P_{c00|0} + \left(1 - \frac{\pi_{c00}}{p_{000} - p_{001}}\right) P_{c01|0}, \tag{6}$$

$$\frac{p_{011}P_{11} - p_{010}P_{01}}{p_{011} - p_{010}} = \frac{\pi_{c00}}{p_{011} - p_{010}} P_{c00|1} + \left(1 - \frac{\pi_{c00}}{p_{011} - p_{010}}\right) P_{c10|1}. \tag{7}$$

Thus, Assumption 3 has the additional testable implications that the left hand sides of equations 6 and 7 must be proper probability distributions, i.e., non-negative everywhere and integrate to 1, since the right hand sides of the same equations are a mixture of two distributions. Applying Proposition 1 of Imai (Forthcoming), whose proof in turn relies on the identification results of Horowitz and Manski (1995) with contaminated data, I derive the sharp bounds for the CATE as well as the Complier Quantile Treatment Effect (CQTE), which is defined as,

$$\tau_{CQTE}(\alpha) \equiv q_{c00|1}(\alpha) - q_{c00|0}(\alpha), \tag{8}$$

where $q_{c00|t}(\alpha) \equiv \inf\{y : P_{c00|t}(y) \geq \alpha\}$ is the $\alpha$-quantile of the distribution $P_{c00|t}$ for $\alpha \in (0, 1)$.

Result 1 Sharp Bounds Under the Exclusion Restriction for Noncompliers. *Let $\Omega$ be the sample space of $Y$. Define the distributions,*

$$Q_0 \equiv \frac{p_{000}P_{00} - p_{001}P_{10}}{p_{000} - p_{001}}, \quad \text{and} \quad Q_1 \equiv \frac{p_{011}P_{11} - p_{010}P_{01}}{p_{011} - p_{010}}.$$

*Also, define* $r_z(\alpha) = \inf\{y : Q_z[-\infty, y] \geq \alpha\}$ *if* $0 < \alpha < 1$, $r_z(\alpha) = \inf\{y : y \in \Omega\}$ *if* $\alpha \leq 0$, *and* $r_z(\alpha) = \sup\{y : y \in \Omega\}$ *if* $\alpha \geq 1$, *for* $z = 0, 1$. *Then, given Assumptions 1, 2, and 3, the following sharp bounds can be derived.*

1. *(Complier Average Treatment Effect)*

$$\tau_{CATE} \in \left[\min_{\pi_{c00}\in\Pi}\left(\int y\, dL_{1-\frac{\pi_{c00}}{p_{011}-p_{010}}|1} - \int y\, dU_{1-\frac{\pi_{c00}}{p_{000}-p_{001}}|0}\right), \max_{\pi_{c00}\in\Pi}\left(\int y\, dU_{1-\frac{\pi_{c00}}{p_{011}-p_{010}}|1} - \int y\, dL_{1-\frac{\pi_{c00}}{p_{000}-p_{001}}|0}\right)\right].$$

*where the distributions,* $L_{\gamma|z}$ *and* $U_{\gamma|z}$ *for* $z = 0, 1$, *are defined as follows,*

$$L_{\gamma|z}[-\infty, y] \equiv \begin{cases} Q_z[-\infty, y]/(1-\gamma) & \text{if} \quad y < r_z(1-\gamma), \\ 1 & \text{if} \quad y \geq r_z(1-\gamma), \end{cases}$$

$$U_{\gamma|z}[-\infty, y] \equiv \begin{cases} 0 & \text{if} \quad y < r_z(\gamma), \\ (Q_z[-\infty, y]-\gamma)/(1-\gamma) & \text{if} \quad y \geq r_z(\gamma), \end{cases}$$

2. *(Complier Quantile Treatment Effect)*

$$\tau_{CQTE} \in \left[\min_{\pi_{c00}\in\Pi}\left\{r_1\left(\frac{\alpha\pi_{c00}}{p_{011}-p_{010}}\right) - r_0\left(1-\frac{(1-\alpha)\pi_{c00}}{p_{000}-p_{001}}\right)\right\}, \max_{\pi_{c00}\in\Pi}\left\{r_1\left(1-\frac{(1-\alpha)\pi_{c00}}{p_{011}-p_{010}}\right) - r_0\left(\frac{\alpha\pi_{c00}}{p_{000}-p_{001}}\right)\right\}\right].$$

The proof is omitted since the result can be proved by using Lemma 1 of Imai (Forthcoming) and applying the argument used in the proof of Proposition 1 of Imai (Forthcoming).

# 4 Additional Assumptions that Tighten the Bounds

Following Zhang and Rubin (2003), I consider two additional assumptions that tighten the bounds derived in Result 1. First, one may assume the following stochastic dominance assumption about the outcome variable in the context of randomized experiments with noncompliance,

Assumption 4 (Stochastic Dominance for Compliers)

$$P_{c00|z}[-\infty, y] \leq P_{c01|z}[-\infty, y], \quad \text{for all } y \in \Omega \text{ and } z = 0, 1$$

Under this assumption, the sharp bounds are available using the following closed-form expressions, Result 2 Sharp Bounds Under the Exclusion Restriction and the Stochastic Dominance. *Suppose that Assumptions 1, 2, 3, and 4. Then the following sharp bounds can be derived.*

1. *(Complier Average Treatment Effect)*

$$\tau_{CATE} \;\in\; \left[\overline{Y}_1 - \int y\, dU_{\frac{p_{110}-p_{111}}{p_{000}-p_{001}}|0},\; \int y\, dU_{\frac{p_{101}-p_{100}}{p_{011}-p_{010}}|1} - \overline{Y}_0\right].$$

2. *(Complier Quantile Treatment Effect)*

$$\tau_{CQTE} \;\in\; \left[r_1(\alpha) - r_0\left(\alpha - \frac{(1-\alpha)(p_{110}-p_{111})}{p_{000}-p_{001}}\right),\; r_1\left(\alpha - \frac{(1-\alpha)(p_{101}-p_{100})}{p_{011}-p_{010}}\right) - r_0(\alpha)\right].$$

The proof follows from a straightforward application of Lemma 2 of Imai (Forthcoming) and hence is omitted.

Second, I consider the following monotonicity assumption,

Assumption 5 (Monotonicity for the Truncation)

$$W_i(1) \;\le\; W_i(0) \quad \text{for all } i.$$

The assumption implies $\pi_{c01} = \pi_{n01} = \pi_{a01} = 0$. Appendix A.2 proves that under this assumption alone, the sharp bounds on $\pi_{c00}$ is given by,

$$\pi_{c00} \;\in\; \Pi^* \;\equiv\; (0,1] \bigcap \left[p_{000} - p_{001},\; \min(p_{000}, p_{011} - p_{010}, p_{000} + p_{100} - p_{001} - p_{101})\right], \qquad (9)$$

with the following testable implications,

$$p_{100} \;\ge\; p_{101}, \quad \text{and} \quad p_{011} \ge p_{010}. \qquad (10)$$

When compared with the sharp bounds under Assumption 3, the upper bound in equation 9 is less informative than that in equation 4. Given these bounds, the sharp bounds on the quantities of interest can be obtained by replacing $\Pi$ in Result 1 with $\Pi^*$ given in equation 9.

If both Assumptions 3 and 5 are made, then $\pi_{c00}$ is identified as,

$$\pi_{c00} \;=\; p_{000} - p_{001}. \qquad (11)$$

Then, the closed-form expression for the sharp bounds are available by simply substituting equation 11 into the sharp bounds in Result 1,

$$\tau_{CATE} \in \left[\int y \, dL_{1-\frac{p_{000}-p_{001}}{p_{011}-p_{010}}|1} - \overline{Y}_0, \int y \, dU_{1-\frac{p_{000}-p_{001}}{p_{011}-p_{010}}|1} - \overline{Y}_0\right], \tag{12}$$

$$\tau_{CQTE} \in \left[r_1\left(\frac{\alpha(p_{000}-p_{001})}{p_{011}-p_{010}}\right) - r_0(\alpha), \; r_1\left(1 - \frac{(1-\alpha)(p_{000}-p_{001})}{p_{011}-p_{010}}\right) - r_0(\alpha)\right]. \tag{13}$$

Finally, if one assumes that Assumptions 3, 4, and 5 hold, then the sharp bounds are further simplified as,

$$\tau_{CATE} \in \left[\overline{Y}_1 - \overline{Y}_0, \int y \, dU_{1-\frac{p_{000}-p_{001}}{p_{011}-p_{010}}|1} - \overline{Y}_0\right], \tag{14}$$

$$\tau_{CQTE} \in \left[r_1(\alpha) - r_0(\alpha), \; r_1\left(1 - \frac{(1-\alpha)(p_{000}-p_{001})}{p_{011}-p_{010}}\right) - r_0(\alpha)\right]. \tag{15}$$

## 5 Concluding Remarks

In this paper, I generalize the results of Zhang and Rubin (2003) and Imai (Forthcoming) to randomized experiments with noncompliance by deriving the bounds on the average and quantile treatment effects for compliers. An alternative approach is to consider the point identification of these treatment effects via parametric modeling (Mattei and Mealli, 2007). The analysis developed in this paper complements such an approach by establishing the identification region without modeling assumptions.

# A Derivation of the Sharp Bounds on $\pi_{c00}$

## A.1 Under Assumption 3

Assumption 3 implies that $\pi_{a00} = p_{010}$, $\pi_{a11} = p_{110}$, $\pi_{n00} = p_{001}$, and $\pi_{n11} = p_{101}$, which in turn yield the following restrictions,

$$\pi_{c01} + \pi_{c00} = p_{000} - p_{001}, \tag{16}$$

$$\pi_{c10} + \pi_{c00} = p_{011} - p_{010}, \tag{17}$$

$$\pi_{c10} + \pi_{c11} = p_{100} - p_{101}, \tag{18}$$

$$\pi_{c01} + \pi_{c11} = p_{111} - p_{110}, \tag{19}$$

which together imply the following four testable restrictions; $p_{000} \geq p_{001}$, $p_{011} \geq p_{010}$, $p_{100} \geq p_{101}$, and $p_{111} \geq p_{110}$. For the rest of the derivation, we assume that these restrictions are met. If the observed data provides a strong evidence against them, then Assumption 3 may not be appropriate. Noting that equation 19 is redundant (can be expressed as a linear combination of the other three equations), the devivation of the sharp bounds on $\pi_{c00}$ is a linear programming problem subject to the equality constraints in equations 16 to 18 and the inequality constraints, $\pi_{stz} \in [0,1]$ for all $s \in \{c, a, n\}$ and $t, z \in \{0, 1\}$. We first enumerate all basic solutions of the implied polyhedron. Define a vector $b = (p_{000} - p_{001}, p_{011} - p_{010}, p_{100} - p_{101})$. Then, there are four basic solutions $(\pi_{c00}, \pi_{c01}, \pi_{c10}, \pi_{c11})$; $(b_2 - b_3, b_1 - b_2 + b_3, b_3, 0)$, $(b_2, b_1 - b_2, 0, b_3)$, $(b_1, 0, b_2 - b_1, b_1 - b_2 + b_3)$, and $(0, b_1, b_2, b_3 - b_2)$. Next, note that each of these basic solutions is feasible only when the following condition holds; $b_2 \geq b_3$, $b_1 \geq b_2$, $b_2 \geq b_1$, and $b_3 \geq b_2$, respectively. This is because $b_1 - b_2 + b_3 \geq 0$ always holds under Assumption 3. Finally, putting together, the bounds are given by,

$$\max(0, p_{011} + p_{101} - p_{010} - p_{100}) \leq \pi_{c00} \leq \min(p_{011} - p_{010}, p_{000} - p_{001}).$$

□

### A.2 Under Assumption 5

Assumption 5 implies that $\pi_{a00} = p_{010}$ and $\pi_{n11} = p_{101}$, which in turn yield the following linearly independent restrictions,

$$\pi_{n00} + \pi_{c00} = p_{000}, \tag{20}$$

$$\pi_{n10} + \pi_{c10} + \pi_{c11} = p_{100} - p_{101}, \tag{21}$$

$$\pi_{a10} + \pi_{a11} = p_{110}, \tag{22}$$

$$\pi_{n00} + \pi_{n10} = p_{001}, \tag{23}$$

$$\pi_{a10} + \pi_{c00} + \pi_{c10} = p_{011} - p_{010}, \tag{24}$$

Equations 21 and 24 give the two testable implications, $p_{100} - p_{101} \geq 0$ and $p_{011} - p_{010} \geq 0$. We solve this linear programming problem by enumerating all basic feasible solutions. There exist 15 such solutions, and they correspond to six unique values of $\pi_{c00}$, i.e., $(0, p_{000}, p_{000} - p_{001}, p_{000} + p_{100} - p_{101}, p_{011} - p_{010}, p_{011} + p_{110} - p_{010})$. This leads to the desired sharp bounds on $\pi_{c00}$. □

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